The Changing Picture of Nonlinearity in Musical Instruments: Modeling and Simulation

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1 Introduction: A Standard Musical Instrument Model

A natural starting point for the study of any physical system is linearisation—leading to great simplification is terms of analysis, and also, in the computer age, to design flexibility and algorithmic simplification in simulation. The acoustics of musical instruments is no exception. One question, then, is: how much of the behaviour of a given instrument can be linearised? The only clear answer is: definitely not all of it. The production of musical sound by an instrument, whether it is struck, blown, or bowed, relies critically on a nonlinear excitation mechanism. One standard model of the musical instrument, then, relies on a subdivision of the instrument into a nonlinear excitation mechanism, which is to a good approximation lumped, and a linear resonator which is distributed, and characterized by a number of natural frequencies, or modes. Such a model has been employed, for many particular cases, for some time—a powerful unified picture emerged, however, with the article by McIntyre, Schumacher and Woodhouse [1]. See Figure 1. Such a characterisation has been enormously useful, not only in investigations in musical acoustics, but also as a means of arriving at efficient synthesis methods, using modal representations [2, 3], methods based on transfer function descriptions [4], or to spectacular effect for certain systems in 1D when a traveling wave formulation is available, leading to the digital waveguide formalism [5, 6].

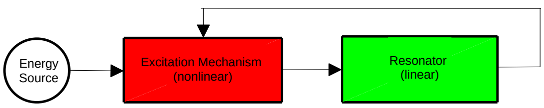

Figure 1: A diagram representing the constituent parts of a standard musical instrument model [1].

Yet, over the past 20 years, the view of the role of the resonator has slowly shifted, at different rates, for all musical instrument types, to include nonlinear effects. In many cases the modifications are minor, leading ultimately to slight differences in timbre, or changes in pitch—but in others, they are dominant, and a linear model of the resonator is grossly insufficient to capture the perceptually salient features of the instrument sound. Linked to the introduction of such nonlinearities in the resonator is the loss of many useful simulation techniques based on linear system theory (though under weakly nonlinear conditions, some interesting extensions are available, as will be indicated). Nevertheless, simulation research has proceeded apace; one interesting unifying concept underlying many new developments has been the notion of passivity, or the maintenance of an energy balance—when transferred to a discrete time algorithm, such a concept leads to robust and flexible algorithm designs.

This paper is intended as a non-technical review of of some of the interesting and relatively new developments in research into resonator nonlinearities in a wide variety of musical instruments. Nonlinear string vibration is covered first, in Section 2, and then the natural extension to the vibration of thin plate structures in Section 3. Shock wave formation in acoustic tubes is discussed in Section 4, and next the exotic and very new area of distributed collision between musical instrument components is briefly outlined in Section 5. Finally, in Section 6, some very general perspectives on the use of passivity concepts in simulation are presented.

2 Strings

Linear string vibration, particularly in the case of motion in one transverse polarisation, and including effects of bending stiffness and loss, has served as a useful starting point for many investigations in musical acoustics [7, 8, 9] and is also extensively used in synthesis [6]. Nonlinear models of string vibration have a long history—the first models can be attributed to Kirchhoff [10] and Carrier [11], and involve a very rough approximation to the interaction between transverse and longitudinal motion—in fact, longitudinal motion is not explicitly included in such models, and its effects (which may be viewed in terms of either an increase on string length, or an increase in string tension) are included as an amplitude-dependent correction to the global wave speed. The primary effect of the use of such a model, then, is an increase in pitch with vibration amplitude—or, when losses are present, to downward pitch glides as amplitude decreases, as in the case of a pluck or strike. See Figure 2. Such models, after lengthy investigations by various authors outside of musical acoustics [12, 13, 14] were employed in musical acoustics studies by Legge and Fletcher [15] and, for two transverse polarisations, by Gough [16]. In the synthesis setting, such effects are sometimes referred to as ”tension modulation,” and have been employed in digital waveguide [17, 18], Volterra series-based [19, 20] and finite difference [21] models of high-amplitude string vibration.

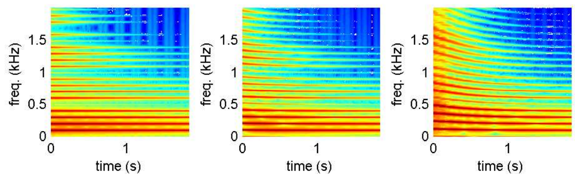

Figure 2: Spectrograms of sound output, for a plucked string under increasing excitation amplitudes, exhibiting typical pitch glide effects.

The tension modulation nonlinearity does allow for characteristic changes in string pitch with amplitude—but more subtle audible effects require a complete modeling of the coupled longitudinal/transverse system. In this case, the longitudinal dynamics are no longer averaged away, but coupled, pointwise, to the transverse motion. Though the effects of longitudinal vibration in strings had been examined previously (see, e.g., [22]), nonlinear ”mixing” of transverse and longitudinal vibration was later identified as a source of so-called phantom partials in strings vibrating at high amplitudes by Conklin [23, 24]. See Figure 3. A model of such nonlinear coupled vibration had long been available; see, e.g., the concise treatment in Morse and Ingard [25]. Such a model was later employed by Bank and

Sujbert [26, 27] in a simplified form as a starting point for a variety of synthesis techniques allowing the emulation of phantom partials in piano tones. Even more recent work has concentrated on complete physical models of the grand piano, incorporating such longitudinal/transverse string models [28].

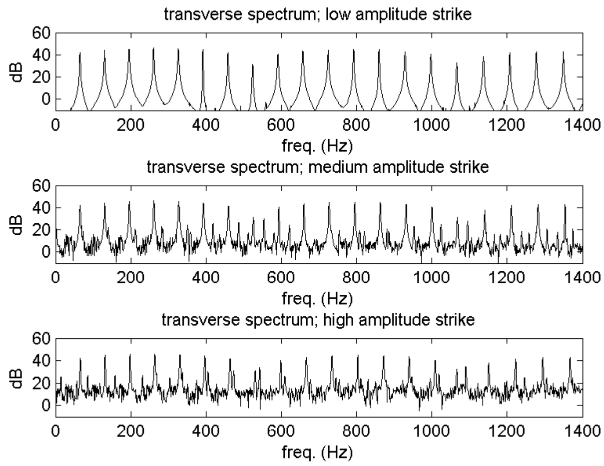

Figure 3: Spectra for lossless string vibration, under striking conditions of increasing strength, exhibiting the appearance of phantom partials.

3 Plates and Membranes

Perhaps the strongest distributed nonlinearity in musical acoustics is that occurring in thin flat structures, which play the role of the resonator in instruments such as cymbals and gongs. When the vibration amplitude is large compared to the thickness, a linear model is grossly insufficient to characterize the behaviour of the instrument. Various characteristic features, including the spontaneous generation of modes, and the dramatic migration of energy towards high frequencies were examined, from an experimental and phenomenological point of view, by various authors (and especially Rossing [29] and Legge and Fletcher [30]).

As in the case of strings, various models are available. A direct generalisation of the ”tension modulated” string model, where longitudinal effects are averaged to yield an effective change in tension, is that due to Berger [31], which has been used in modal-based synthesis methods for drums [32]. In the case of strings, the use of more complex models leads to relatively subtle effects (such as, e.g., phantom partials). In the case of thin plates, however, such models are essential. The use of the Föppl-von Kármán model [33] for vibration at moderate amplitudes to explain such phenomena in musical instruments is relatively recent—see, e.g., Touzé et al. [34], and has opened the way towards simulation methods for such objects, through methods such as finite difference schemes [35], and also modal approaches [36], and may be extended to the case of curved plates (or shells) [37, 38] in order to model instruments such as cymbals. See Figure 4, illustrating the spontaneous generation of higher frequencies given a smooth initial shape, and a higher gross rate of vibration. Needless to say, computational costs associated with such simulations are extreme, regardless of the method employed.

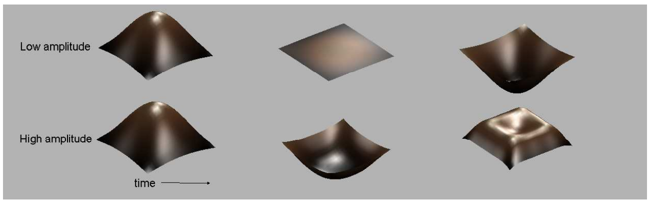

Figure 4: Time evolution of a square plate, initialised to its first linear mode shape (under simply supported conditions) under linear conditions (at top), and nonlinear conditions (at bottom).

Interestingly, recent simulation and experimental work [39, 40] indicates that such effects are also at work in drums as well (where tension, rather than stiffness is the main restoring force)—though not leading to the same degree of departure from the linear model, it is clear that the bright and noisy timbres of instruments such as toms or bass drums when struck at high amplitudes are dependent on such nonlinear effects.

4 Acoustic Tubes

The standard linear model of wave propagation in a wind instrument is invariably a descendant of the model of Webster [41], which models one-dimensional wave propagation in a tube of variable cross-section, and terminated by a condition modeling radiation—for tubes of small cross-section, viscothermal wall losses play a non-negligible role in determining the widths of the impedance peaks in the spectrum, which in turn is strongly related to playability, particularly in brass instruments. A variety of such models are available, generally posed in the frequency domain in terms of impedance and admittance—see, e.g., [42, 43, 44].

At high amplitudes, however, it is now generally accepted that nonlinear steepening effects occur along cylindrical segments of the bore. See Figure 5. Such work was initiated by Hirschberg et al. [45], after earlier observations by Beauchamp—see [46] for a review. Synthesis applications were developed by Msallam et al. [47], and Vergez and collaborators [48]. Such steepening effects are intimately related to ”brassy” timbres at high blowing pressures [49, 50].

Numerical modeling in this case is a difficult challenge—though numerical techniques for shock capturing of course have a very long history (see the early review by Sod [51]), the difficulty in the musical setting is to design a method such that the solution is not distorted, perceptually, as may be the case for certain commonly used techniques (such as, e.g., artificial viscosity [52]). This difficulty is alleviated somewhat by the relatively weak strength of the shocks which form (pressure deviations in a brass instrument rarely exceed 10% of atmospheric). A greater difficulty is the complexity of the model, particularly when variation of the bore profile and viscothermal wall losses are taken into account. Partial models are available in this case—see, e.g., [53], for the case of lossless ducts. Various simulation strategies have been proposed: harmonic balance techniques are discussed in [54], and time-stepping methods, for unidirectional waves in [55]. In general, finite volume methods [56] would appear to be well-suited to this particular problem.

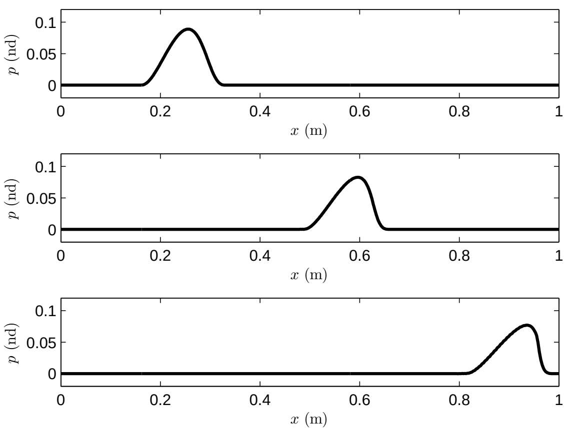

Figure 5: Snapshots of the time evolution of a traveling pulse in a cylindrical tube, illustrating shock wave formation.

5 Distributed Collisions

Collisions play an obvious role in keyboard and percussion instruments, in which case a hammer [8, 9, 57] or mallet [58, 59], usually modelled as lumped, comes into contact with a resonating body such as a string, bar, membrane or plate. The interaction is nonlinear, and often modelled using a variant of Hertz's law of contact (or a power-law nonlinearity in the compression of the striking object), perhaps including effects of loss, as per the model of Hunt and Crossley [60]. For such interactions, the main effect is that of the reduction in contact time (which is generally quite short, and on the order of 1-10 ms for most instruments) with striking velocity, leading to a perceived brightening of timbre.

But it is clear that there is a wide variety of other situations in which collisions play an important role, and in which one or both of the objects in contact must be modelled as distributed. A basic example is the interaction of a string in free vibration against a rigid barrier [61], as in the case of the sitar or timbura [62, 63, 64]. See Figure 6. In these cases, beyond a brightening of timbre, because the contact region is distributed there is a time-varying change in timbre, sometimes accompanied by changes in pitch if the effective length of the string is shortened at high amplitudes. Other examples include the pinning of a string against a barrier by a finger as in the case of the bowed string family [65], and also against more elaborate barriers such as the fretboard in the case of the guitar [66]. Perhaps the most dramatic example of all is that of the snare drum [67], in which case a multitude of distributed wires are in partial contact with a membrane, which must also be modelled as distributed.

The collision interaction in these cases is far from linear—and furthermore, the nonlinearity is not even approximately smooth, in contrast to the case of inherent nonlinearities in strings and plates. Such systems have been approached occasionally in synthesis applications [68, 69]; see [70] for some recent numerical work on a variety of musical systems involving collisions.

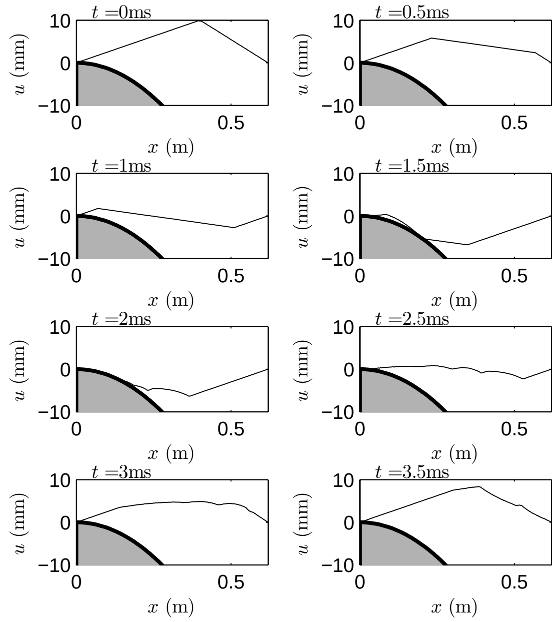

Figure 6: Snapshots of the time evolution of a string colliding with a rigid parabolic barrier.

6 Concluding remarks: Passive Representations and Numerical Methods

Though there is not space in this short review for a full look at numerical methods for distributed nonlinear systems, it is worth taking a look at the concept of passive systems is miniature, as such representations form a solid design strategy for various nonlinear systems in musical acoustics; such representations are heavily used in various different guises in acoustics simulations, and particularly in sound synthesis—scattering structures such as digital waveguides [6], as well as wave digital filters [71] all employ such concepts (leading to structures based on the ue of delay lines or shifts, and norm-preserving operations such as adaptors or scattering junctions). Non-wave based methods may also be written in a passive form, as in the case of, e.g., finite difference schemes [72], or other newer formalisms such as port-Hamiltonian methods [73] applied to musical systems [74]. of course, outside of musical acoustics, such methods have a long history in mainstream simulation, in the form of Hamiltonian integrators [75, 76, 77].

A general passive system obeys a power balance of the form

$$\frac{dH}{dt} = -Q + P \tag{1}$$

where here, $H = H(t) \geq 0$ is the total stored system energy, $Q(t) \geq 0$ is power loss, and $P(t)$ is input power. When the system is lossless and unforced, it is often referred to as a Hamiltonian system, and $H(t)$ (the Hamiltonian) itself is non-negative and conserved. $H(t)$ itself may be broken down as $H = T + V$, where $T(t)$ is the system kinetic energy (almost always a positive definite quadratic form in the system velocities), and the potential $V(t)$, which, for nonlinear systems, is usually not a quadratic form, but which remains non-negative.

If the non-negativity of H and the loss term Q may be transferred to discrete time, in such a way that the power balance is preserved from one time step to the next, i.e., as

$$\frac{1}{k}\left(H^{n+1} - H^{n}\right) = -Q^{n} + P^{n} \tag{2}$$

where now, H^n, Q^n and P^n are time series indexed by integer n, and where k is a time step (possibly varying, but usually fixed to a given sample rate in acoustics and synthesis applications), then such a representation becomes a useful means of bounding of solutions, leading to numerical stability conditions.

For musical systems, though, some more care is required. Usually, the system of interest can be linearised—leaving aside the question of nonlinear loss for the moment, this is general implies a decomposition of the potential energy as

$$V = V_{linear} + V_{nonlinear} \tag{3}$$

where V_{linear} is necessarily a quadratic form. Ideally, one would like to be able to treat these two terms separately, so as to be able to design a fine-grained design for the linear part of the system, and then add in nonlinear effects as a refinement. This is most straightforward if the linear and nonlinear potential energy components are separately non-negative. In other words, the nonlinearity is of a "hardening" variety. While this is true for some systems (such as, e.g., the Kirchhoff-Carrier system, the Föppl-von Kármán system, and some collision models), it is not true for others such as the system describing coupled longitudinal-transverse motion of a string. Coming up with an efficient design under these conditions then becomes a much more difficult problem, and constitutes a major design challenge. See Figure 7.

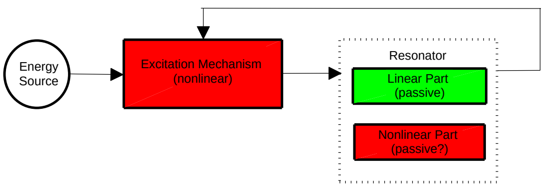

Figure 7: A revised diagram representing the constituent parts of a standard musical instrument model, with a subdivision of the resonator into its linear and nonlinear parts.

Acknowledgments

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