Nonlinear Effects in Drum Membranes

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The linear behaviour of drum membranes has been extensively studied and is now well understood. Particular attention has been devoted to modal analyses of circular drum heads, and good agreement has been found between experiment and theory. Up to now, however, there has been relatively little investigation into the relevance and nature of nonlinear effects in drum membranes.
Stiff strings and stiff plates, however, have seen a good deal of such investigation: pitch glides and the migration of energy towards higher frequencies are typical phenomena that can be found at high striking amplitudes. Such effects result from nonlinearities that arise due to coupling between transverse and longitudinal wave motion in the material.
It has recently been shown that nonlinear effects, similar to those encountered in stiff strings and plates, can be important for drum membranes, both from a physical and a perceptual point of view. While existing tension-modulation techniques provide a useful starting point for modelling the effects of these nonlinearities, a complete nonlinear model is required for a more accurate, and ultimately more realistic, description.
In this study a nonlinear finite difference time domain model of a tom-tom is used, alongside experimental evidence, to highlight and quantify the relevance of nonlinear phenomena in drum membranes. The model includes geometrical membrane nonlinearities, and full air coupling between the two drum membranes. Experimental evidence is obtained from measurements of internal and external sound pressure fields around the drum.

1 Introduction

The motion of a circular membrane is well understood in terms of modes of the 2D wave equation. When dealing with real drumheads, however, additional phenomena must be taken into account, like bending stiffness and air loading effects, which alter the frequency of the modes in a vacuum [1]. Although single membrane drums do exist (like the rototom), it is far more common for these instruments to have a cavity (as in the case of a kettledrum), and often an additional resonant membrane (like tom toms or bass drums). The theoretical calculation of modes and frequencies for these systems becomes obviously more involved as the number of interacting components increases. In these cases, physics-based numerical simulation [2] is a viable approach for the analysis of the behaviour of the instrument.

An interesting effect that can be noticed in real drums is the pitch glide. At high striking amplitudes, the perceived pitch immediately after the blow gradually settles to a lower frequency. It is clear that a simple mistuning or inhomogeneous tension of the membrane cannot account for such nonlinear behaviour [3], as it would simply cause a frequency splitting of the degenerate linear modes of the membrane [4]. Experimental evidence for this phenomenon has been reported, for example, in [5], where a frequency shift of 10% in the first mode was observed on a bass drum.

A relatively simple theoretical model which is able to describe tension modulation effects is Berger's [6]. Such physical description, originally proposed in the framework of large deflections of plates, is similar to the Kirchhoff-Carrier model [7] for string nonlinearities, and is indeed able to produce pitch glides. It has been recently adopted for the simulation of single membranes [8] and timpani drums [9]. A similar model with a slightly different mathematical expression can be found in [10] and has been explored in the context of sound synthesis.

In a recent work [11], it has been questioned whether such a simplified tension modulation model is enough to describe the nonlinear effects involved, or if a more complex model must be taken into account. This issue will be investigated in the present paper, where experimental results will be compared with numerical simulation. This paper is organised as follows: the experimental setup will be described in Section 2, while a brief outline of the numerical code employed will be given in Section 3. Finally, results of both experiment and simulation will be presented and discussed in Section 4.

2 Experimental setup

A preliminary experimental investigation has been conducted on a floor tom with radius $R = 20$ cm and height 42 cm. The thickness was 0.175 mm and 0.19 mm for the upper and lower membrane, respectively. The wave speed for each membrane has been measured indirectly by lightly tapping the drumhead and fitting the first peaks of the Fourier spectrum with the modes of an ideal membrane. Though not extremely accurate, given the discussion in the previous section, these measurements provide an estimate of the relevant parameters of the drum to be used in numerical simulations (see below.) The drum was positioned with drumheads aligned vertically above an even surface and held still during the strikes by placing it over thin rubber strips, to prevent it from moving. A laser doppler vibrometer (LDV; Polytec OVF-5000) was pointed at a small reflecting patch attached to the membrane in order to measure the displacement of the membrane. A near field microphone (Bruel & Kjaer Type 4134) was positioned a few centimetres from the membrane, at half a radius of distance from the centre. Data acquisition was performed with a Bruel & Kjaer PULSE system connected to a PC, which allowed the syncronous recording of all the signals. The upper membrane of the drum was excited with a drum stick. Though a mechanical striking device was not available, care has been taken in order to maintain similar position and amplitude across the various strikes. This can be confirmed *a posteriori* by checking the coherency of the various spectrograms.

3 Numerical Simulation

From a numerical simulation point of view, the drum under consideration can be modelled as a set of two circular membranes, connected by a rigid shell and immersed in a finite box of air $\mathcal{V}$ (see Figure 1.) Each membrane can be modelled as a stiff object, with nonlinearities described by either Berger or von Kármán equations (see Section 3.2.) The numerical implementation of these two models as been discussed in [9] for timpani drums and in [11] for double

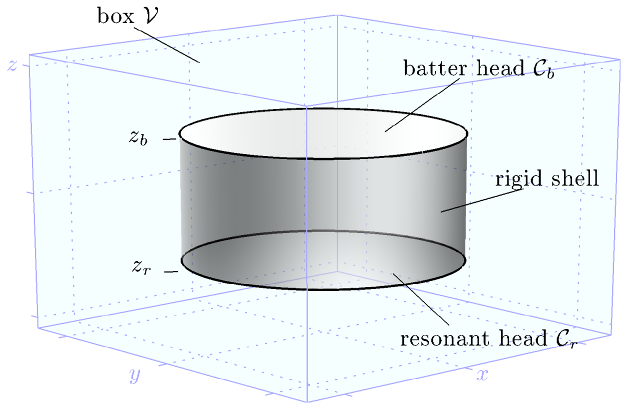

Figure 1: Geometry of the numerical model.

Membranes	
T	tension (N/m)
ρ	surface density (kg/m^2)
E	Young's modulus (N/m^2)
H	thickness (m)
ν	Poisson's ratio
σ_0	frequency independent loss coefficient (1/s)
σ_1	frequency dependent loss coefficient (m^2/s)
A	area of the membrane (m^2)
Air	
c_a	wave speed (m/s)
σ_a	viscothermal loss coefficient (m)

Table 1: *List of physical parameters used in this model.*

membrane drums, respectively, so many of the details will be omitted. Only the underlying physical model will be presented here.

3.1 Description of the model

Let C be a circular region of radius R at vertical position z, and let $w(x, y, t)$ represent the displacement of the batter (upper) membrane at time t and position $(x, y) \in C$. The equation of motion for w can be written as:

$$\rho\partial_{tt}w = \mathcal{L}[w] + \mathcal{N} + \mathcal{E}. \tag{1}$$

$\mathcal{L}$ groups together all the linear terms (wave propagation, stiffness, losses),

$$\mathcal{L}[w] = \left(T\Delta_{2D} - D\Delta_{2D}^2 - 2\rho\sigma_0\partial_t + 2\sigma_1\partial_t\Delta_{2D}\right)w, \tag{2}$$

where Δ_{2D} and Δ_{2D}^2 are the Laplacian and biharmonic operators, respectively, and all physical parameters are listed in Table 1. $\mathcal{N}$ represents the nonlinear term and $\mathcal{E}$ represents all the external forcing terms, including excitation and air coupling. The excitation consists of a raised cosine impulse injected into the upper membrane. Similar equations hold for the displacement of the resonant membrane.

The air surrounding the drum can be modelled by means of a velocity potential $\Psi(x, y, z, t)$ at position (x, y, z) within the box, obeying the 3D wave equation with viscothermal losses [12]:

$$\partial_{tt}\Psi = c_a^2\Delta_{3D}\Psi + c_a\sigma_a\partial_t\Delta_{3D}\Psi, \tag{3}$$

where Δ_{3D} is the 3D Laplacian operator. The inclusion of a loss term in the equation is necessary in implementation to remove some numerical artifacts related to the dispersion of the 3D scheme [13] (see [14] for details.)

At the walls of the box, absorbing conditions are applied, while reflective conditions are applied over the cylindrical enclosure of the drum, in order to simulate the cavity. At the interface with the two membranes, coupling conditions must be enforced. In particular, the velocity of the membrane must be equal to the velocity of the surrounding air, which in turn exherts a pressure above and below the membrane itself. An explicit expression for these coupling conditions can be found in [9, 11].

The numerical implementation of this model relies on the discretisation of the physical equations of the system using the finite difference method [15]. Stability of the scheme is a major concern in this kind of simulations, but it can be successfully addressed and guaranteed via energy methods. It can be shown, in fact, that a numerical energy for the scheme exists, which is conserved to machine accuracy in the lossless case [11].

3.2 Membrane nonlinearity

As mentioned above, the form of the nonlinear term can be chosen in different ways. The simplest nonlinear model is probably Berger's [6], and can be written as:

$$\mathcal{N}^B[w] = \frac{EH}{2A(1-\nu^2)}\left(\int_C |\nabla w|^2\right)\Delta_{2D}w \equiv \bar{T}(t)\Delta_{2D}w. \tag{4}$$

This term formally introduces in the system an additional tension $\bar{T}(t)$ which can be expressed as a quasistatic part plus a (lower amplitude) oscillating component [8]. The influence of this term is more important soon after the strike, and gradually disappears as the energy of the membrane is dissipated.

This is but a simplified version of the Föppl-von Kármán system of equations, which is often used in thin plate theory [16] to describe the regime of geometric nonlinearities (nonlinearities due to rotations in the medium, while stress-strain relations in the material are assumed to be linear.) At high vibration amplitudes, this model is able to produce pitch glide effects and crashes, which are typical of gongs and cymbals [17]. The use of this type of nonlinearity for thin membranes is discussed in detail in [18, 19]. This model can be defined as follows:

$$\mathcal{N}^K[w, F] = \mathcal{K}[w, F] \tag{5}$$

where the von Kármán operator $\mathcal{N}^K$ now depends also on the Airy's stress function F defined as

$$\Delta_{2D}^2 F = -\frac{EH}{2}\mathcal{K}[w, w], \tag{6}$$

with $\mathcal{K}[f, g] = \partial_{xx}f\partial_{yy}g + \partial_{xx}f\partial_{yy}g - 2\partial_{xy}f\partial_{xy}g$.

At low vibration amplitudes, both nonlinear terms have negligible influence, and the system is effectively linear. At even moderate amplitudes, however, they both produce pitch glide effects, but they exhibit a different behaviour, which will be analysed in the following sections.

4 Results

4.1 Experiment

In this section we discuss the results from the experiments. The upper membrane was struck with a drumstick at 5 cm from the centre with increasing striking amplitudes. Several repetitions at similar intensity were captured.

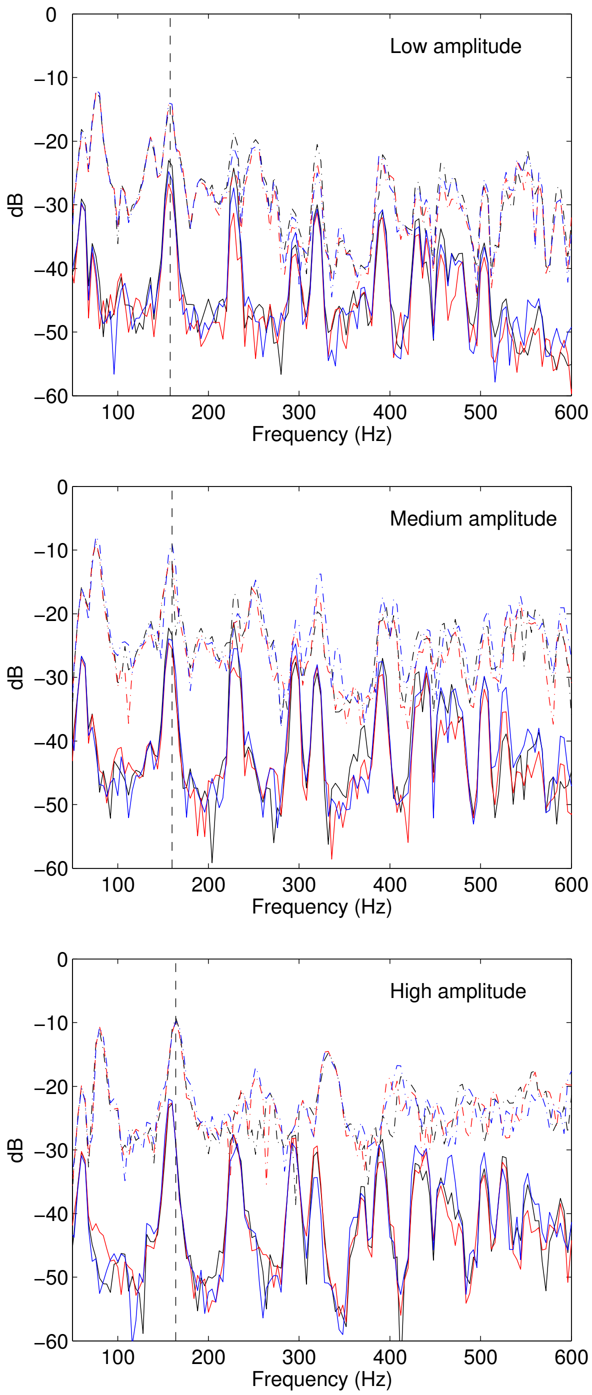

Figure 2: Spectrum of a portion of 0.25 s of the microphone output for three different repetitions (in black, red and blue, respectively) at $t = 0$ s (dashed) and $t = 1$ s (solid line). Three different striking amplitudes are represented.

Figure 2 presents spectra for the sounds recorded with the microphone at three different striking amplitudes: low, medium and high. Fourier transforms are calculated over a portion of approximately 0.25 s (16384 samples at acquisition sampling rate 65536 Hz) at the beginning of the signal ($t = 0$ s) and at $t = 1$ s. They are represented in dashed and solid lines in the upper and lower part of each picture, respectively. Three different repetitions are analysed for each striking amplitude (represented in black, red and blue), for a total of 9 sounds. Even if strikes were injected with a normal drumstick by the authors, care has been taken to maintain the same position and force within each set of data, and this can be confirmed *a posteriori* by verifying the coherence of the various spectra.

At low striking amplitudes, the various peaks simply decay in intensity with different rates, but do not change frequency. This a typical linear behaviour, and is associated with small vibration amplitudes of the membrane. At medium amplitudes (second picture), the peaks of the initial portion of the signal have a slightly higher frequency than those in the later part, which signifies that nonlinear phenomena start to be present, and a limited pitch glide starts to be noticed. This effect becomes evident at high striking amplitudes (third picture), and a pitch glide can clearly be heard. It is possible to concentrate the attention on a single peak (marked with a vertical dashed line), for example the one corresponding to the (1,1) mode with membranes in phase, which is the one that resonates longer [20]. For low amplitudes, the frequency of this peak for the initial portion of the signal is 158 Hz. This frequency becomes 160 Hz at medium amplitudes, and 164 Hz at high amplitudes. In all three cases, the frequency of this mode settles to 158 Hz in the later part of the sound, with a relative shift of 22 and 65 cents for medium and high amplitudes, respectively.

In order to quantify the displacement of the membrane in the various cases, it is possible to analyse the vibrometer signal recorded for the central point of the drumhead (see Figure 3). As expected, the displacement of the membrane is proportional to the striking force. At low amplitudes, the maximum displacement recorder was 2 mm, at medium amplitudes this was slightly less than 4 mm, while for hard strikes the drumhead movement was more than 5 mm.

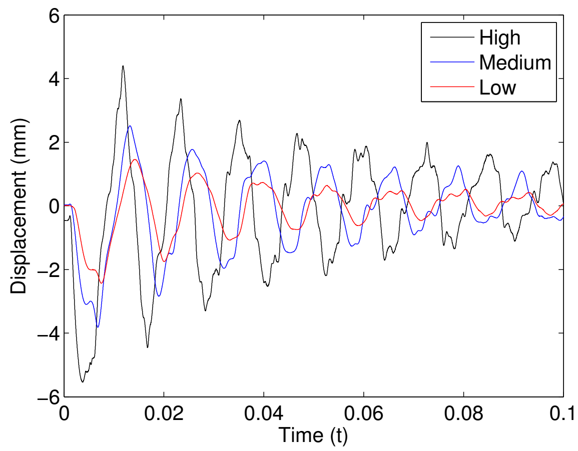

Figure 3: Experimental values for the displacement of the central point of the drumhead at different striking amplitudes (colors as in the legend).

4.2 Numerical simulation

In order to compare the behaviour of the nonlinear models presented in Section 3.2, it is possible to run numerical simulations with identical physical parameters. The virtual drum has height 42 cm, with identical mylar membranes of radius $R = 20$ cm, thickness $H = 0.175$ mm, density $\rho = 2600$ kg/m^3 and tension $T = 1577$ N/m. The excitation is injected as a raised cosine function with duration 3 ms into the upper membrane, half way between the centre and the rim. The maximum displacement of the membranes in both cases was 5 mm, compatible with those found in the experiment. Figure 4 shows the spectrograms of the outputs obtained from the simulations. It is apparent how Berger model produces a wider pitch glide, while more energy migrates towards higher frequencies in the von Kármán case. In order to quantify the pitch glide effect, it is possible to plot a Fourier spectrum of a portion of the signal at different times, in order to show the instantaneous frequency peaks, as done in the previous section. This is shown in Figure 5, where a portion of the signal of 8192 samples (≈ 0.25 s at sample rate 32 kHz) is analysed at $t = 0$ s and $t = 1.0$ s. In both cases, the later portion of the signal looks very similar, confirming that at low amplitudes the two models are identical and reduce to the linear one, as expected. The initial part of the signal, however, shows a remarkable difference, with the peaks for Berger model being significantly higher in frequency. For the second peak, for example, the linear frequency is 152 Hz, with initial frequencies of 158 Hz and 172 Hz for von Kármán and Berger model, respectively, and relative shift of 67 and 214 cents, respectively. The result for von Kármán model is in agreement with the experimental one at high amplitudes. In Berger model, peaks are also wider, signifying that the frequency shift is faster, as can be noticed from the spectrograms in Figure 4. Therefore, from a sound synthesis perspective, von Kármán model seems to give better results in terms of realism during the attack portion of the sound, while both models present the same behaviour at low amplitudes, as expected.

5 Final remarks and future work

In this paper, we analysed the nonlinear effects arising in drum membranes at high striking amplitudes. We presented experimental measures of pitch glides in a floor tom. Then, we introduced finite difference numerical simulations in order to compare two different physical models that describe nonlinear effects in thin stiff structures, Berger and von Kármán model. We showed that Berger model produces a wider pitch glide than von Kármán, which then gives best results when it comes to sound synthesis.

Some aspects need to be addressed in future works in order to improve the present analysis. From the experimental perspective, a sistematic excitation mechanism has to be taken into account, in order to achieve a more controllable setup. Though great care has been taken to keep a constant excitation force, this method does not guarantee a repeatable and quantifiable measure for the striking amplitude. A thourough measure of the physical parameters of the drum needs to be carried out, as well, in order to have the correct parameters to feed into the simulation. From the numerical point of view, a realistic striking mechanism has not been adopted, and a simplified raised cosine excitation has been used, instead. Though a novel, energy conserving scheme for the simulation of the nonlinear collision force between the mallet and the membrane exists [21], it is not straightforward to implement it in the present model, as two strongly nonlinear terms would be present at the same time. This, however, is currently under study.

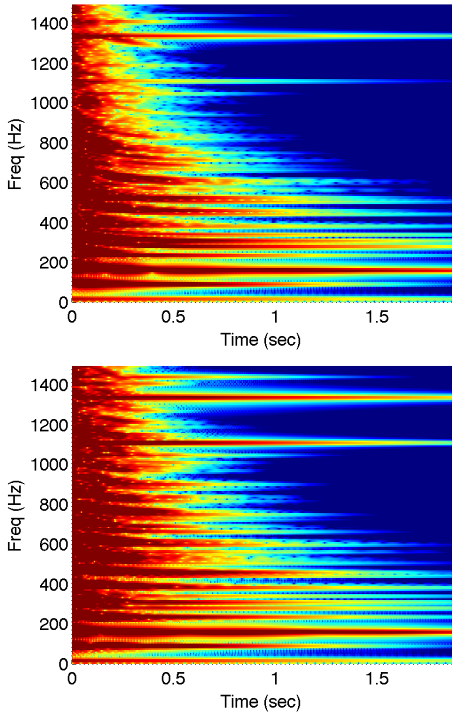

Figure 4: Spectrograms of the simulated outputs at sample rate 32 kHz. Left: Berger model. Right: von Kármán model.

Acknowledgments

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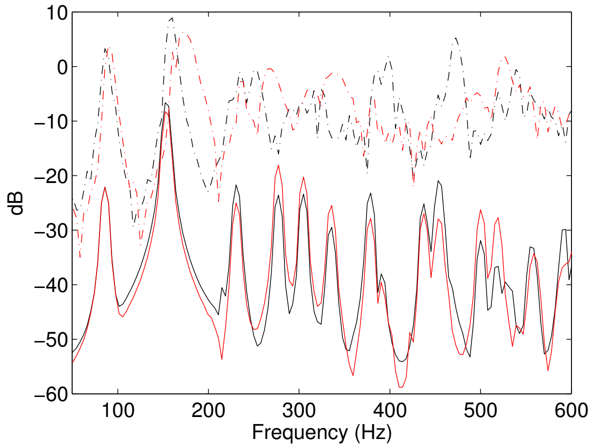

Figure 5: Spectra of the outputs obtained from numerical simulations. Berger model (in red) exhibits a wider pitch glide than von Kármán (in black).

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